

Transversely Lie holomorphic foliations on projective spaces

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Abstract

We prove that a one-dimensional foliation with generic singularities on a projective space, exhibiting a Lie group transverse structure in the complement of some codimension one algebraic subset is logarithmic, i.e., it is the intersection of codimension one foliations given by closed one-forms with simple poles. If there is only one singularity in a suitable affine space, then the foliation is induced by a linear diagonal vector field.

1 Introduction

In the study of foliations it is very useful to consider the transverse structure¹². Among the simplest transverse structures are Lie group transverse structure, homogeneous transverse structure and Riemannian transverse structure. In the present work we consider foliations with a Lie group transverse structure. In few words, this means that the foliation is given by an atlas of submersions taking values on a given Lie group G and with transition maps given by restrictions of left-translations on the group G , called the *transverse model* for \mathcal{F} . We shall refer to such a foliation as a G -foliation. The theory of G -foliations is a well-developed subject and follows the original work of Blumenthal [4]. In codimension one, the Riemann-Koebe uniformization theorem implies that any holomorphic G -foliation is indeed *transversely additive* (i.e., of transverse model the additive group of complex numbers) and therefore given by a closed holomorphic one-form ([16]). Using this and some extension techniques from Holomorphic Foliations it is possible to make an accurate study of codimension one holomorphic foliations on complex projective spaces admitting a Lie group transverse structure on the complement of some invariant algebraic subset (of codimension one in the nontrivial case) ([16], [17]). On the other hand, as for the local situation, it is possible to prove that a nondegenerate isolated singularity of a holomorphic one-form in dimension two, admitting a G -transverse structure in the complement of its set of local separatrices which is assumed to be composed of only finitely many curves, is analytically equivalent to its formal normal form as introduced in [14] and [15]. Our current aim is to motivate the study of the codimension $\ell \geq 2$ case beginning with the more basic situation. More precisely, we study one-dimensional holomorphic foliations on complex projective spaces, with *generic*

¹MSC Classification: 57R30, 22E15, 22E60.

²Keywords: Foliation, Lie transverse structure, fibration.

singularities, and admitting a Lie group transverse structure in the complement of some codimension one algebraic invariant subset.

In codimension one, a very basic example of foliation with generic singularities and with Lie group transverse structure in the complement of some algebraic hypersurface is the class of logarithmic foliations. A *logarithmic foliation* on a projective manifold V is one given by a closed rational one-form η with simple poles. A *Darboux foliation* on V is a logarithmic foliation given by a rational one-form η as follows: $\eta = \sum_{j=1}^r \lambda_j \frac{df_j}{f_j}$, where the f_j are rational functions on V and $\lambda_j \in \mathbb{C} \setminus \{0\}$ (see [17] and [8] for more on logarithmic and Darboux foliations). It is well-known that any logarithmic one-form η in \mathbb{CP}^m is of Darboux type ([16]). We extend these notions in a natural way. By a *logarithmic foliation* of dimension one on a manifold V^m we mean a foliation which is given by a system of $m - 1$ closed meromorphic one-forms $\eta_1, \dots, \eta_{m-1}$, all with simple poles and linearly independent in the complement of their sets of poles. A *Darboux foliation* of dimension one is a logarithmic foliation given by one-forms η_j of Darboux type. An isolated singularity of a holomorphic vector field X is *without resonances* if the eigenvalues of the linear part DX at the singular point, are linearly independent over \mathbb{Q} . Local linearization of such a (non-resonant) singularity is granted if this singularity is in the *Poincaré domain*, i.e., the convex hull of its eigenvalues does not contain the origin. In the other case, i.e., in the *Siegel domain* there are Diophantine conditions that assure local linearization ([6], [18]).

Our main result below is a first step in the comprehension of the possible Lie group and homogeneous transverse structures (see [10]) for codimension $\ell \geq 1$ holomorphic foliations on projective spaces.

Theorem 1. *Let \mathcal{F} be a one-dimensional holomorphic foliation with singularities on \mathbb{CP}^m . Assume that:*

1. *The singularities of \mathcal{F} are linearizable without resonances.*
2. *There is an invariant codimension one algebraic subset $\Lambda \subset \mathbb{CP}^m$ such that \mathcal{F} has a G -transverse structure in $\mathbb{CP}^m \setminus \Lambda$.*

Then \mathcal{F} is a logarithmic foliation.

If moreover $\sharp[\text{sing}(\mathcal{F}) \cap \mathbb{C}^m] = 1$ for an affine space $\mathbb{C}^m \subset \mathbb{CP}^m$ with $\mathbb{CP}^m \setminus \mathbb{C}^m$ in general position with respect to \mathcal{F} , then \mathcal{F} is linearizable, i.e., $\mathcal{F}|_{\mathbb{C}^m}$ is induced by a vector field X that can be put in the linear diagonal form $X = \sum_{j=1}^m a_j x_j \frac{\partial}{\partial x_j}$.

As spoliu of the proof of Theorem 1 we obtain:

Theorem 2. *Let \mathcal{F} be a one-dimensional holomorphic foliation with singularities on a connected complex manifold V^m , of dimension $m \geq 2$, such that:*

1. *There is a codimension one analytic subset $\Lambda \subset V^m$ such that $\mathcal{F}|_{V \setminus \Lambda}$ has a G -transverse structure.*

2. There is a singular point $p \in \text{sing}(\mathcal{F})$ which is linearizable without resonances.

Then:

- (i) $\mathcal{F}|_{V \setminus \Lambda}$ is given by $m - 1$ closed holomorphic one-forms $\eta_1, \dots, \eta_{m-1}$.
- (ii) Necessarily $p \in \Lambda$ and $\eta_1, \dots, \eta_{m-1}$ extend meromorphic with simple poles to a neighborhood of p and therefore to each irreducible component of $\Lambda \setminus \text{sing}(\mathcal{F})$ that intersects V .
- (iii) If each singularity $q \in \text{sing}(\mathcal{F})$ is linearizable without resonance, then \mathcal{F} is a logarithmic foliation indeed, each η_j extends to a meromorphic one-form in V , with simple poles and \mathcal{F} is given by the system $\{\eta_1, \dots, \eta_{m-1}\}$

2 G -foliations

Let \mathcal{F} be a codimension ℓ foliation on a manifold V . Given a Lie group G of dimension ℓ we say that \mathcal{F} admits a *Lie group transverse structure of model G* or a *G -transverse structure* for short, if there are an open cover $V = \bigcup_{j \in J} U_j$ of V such that

on each open set we have defined a submersion $f_j: U_j \rightarrow G$ such that the leaves of $\mathcal{F}|_{U_j}$ are levels of f_j and on each nonempty intersection $U_i \cap U_j \neq \emptyset$ we have $f_i = g_{ij} f_j$ for some locally constant map $g_{ij}: U_i \cap U_j \rightarrow G$. In other words: \mathcal{F} is defined by the submersions $f_j: U_j \rightarrow G$ which on $U_i \cap U_j$ differ by left translations $f_i = L_{g_{ij}}(f_j)$ for some locally constant $g_{ij} \in G$. In this case we call \mathcal{F} a *G -foliation*. The characterization of G -foliations is given by the following result:

Theorem 3 (Darboux-Lie, [10]). *Let \mathcal{F} be a codimension ℓ foliation on V and G a Lie group of dimension ℓ . If \mathcal{F} admits a G -transverse structure then there are one-forms $\Omega_1, \dots, \Omega_\ell$ in V such that:*

- (a) $\{\Omega_1, \dots, \Omega_\ell\}$ is a rank ℓ integrable system which defines \mathcal{F} .
- (b) $d\Omega_k = \sum_{i < j} c_{ij}^k \Omega_i \wedge \Omega_j$ where $\{c_{ij}^k\}$ are the structure constants of a Lie algebra basis of G .

If \mathcal{F}, V and G are complex (holomorphic) then the Ω_j can be taken holomorphic. Given any basis $\omega_1, \dots, \omega_\ell$ of the Lie algebra of G the structure constants $\{c_{ij}^k\}$ of this basis can be obtained above.

Conversely, given a maximal rank system of one-forms $\Omega_1, \dots, \Omega_\ell$ in V such that $d\Omega_k = \sum_{i,j}^k c_{ij}^k \Omega_i \wedge \Omega_j$, where the $\{c_{ij}^k\}$ are the structure constants of the basis $\{\omega_1, \dots, \omega_\ell\}$ of the Lie algebra of G , then:

- (c) For each point $p \in V$ there is a neighborhood $p \in U_p \subset V$ equipped with a submersion $f_p: U_p \rightarrow G$ which defines \mathcal{F} in U_p such that $f_p^{\omega_j} = \Omega_j$ in U_p , for all $j \in \{1, \dots, \ell\}$.

- (d) If V is simply-connected we can take $U_p = V$.
- (e) If $U_p \cap U_q \neq \emptyset$ then in the intersection we have $f_q = L_{g_{pq}}(f_p)$ for some locally constant left translation $L_{g_{pq}}$ in G .

In particular, \mathcal{F} has a G -transverse structure.

3 Examples

The most trivial example of a G -foliation is given by the product foliation on a manifold $V = G \times N$ product of a Lie group G by a manifold N . The leaves of the foliation are of the form $\{g\} \times N$ where $g \in G$. Other basic examples of G -foliations are:

1. Let H be a closed (normal) subgroup of a Lie group G . We consider the action $\Phi: H \times G \rightarrow G$ given by $\Phi(h, g) = h.g$ and the quotient map $\pi: G \rightarrow G/H$ (a fibration) which defines a foliation \mathcal{F} on G . Given $x \in \mathcal{F}_g = \pi^{-1}(Hg)$ we have $\pi(x) = Hg$ and $\Phi_h(x) = h.x$. But $\pi(\Phi_h(x)) = \pi(h.x) = H.hx = Hx$ implies that $\Phi_h(x) \in \pi^{-1}(Hx) = \mathcal{F}_x$ and the orbit $\mathcal{O}(g) = Hg$ is transverse to the fiber $\pi^{-1}(Hg)$. Hence, \mathcal{F} is a foliation invariant under the transverse action Φ . Now let G be a simply-connected group, H a discrete subgroup of G and $F: H \rightarrow \text{Diff}(G)$ the natural representation given by $F(h) = L_h$. The universal covering of G/H is G with projection $\pi: G \rightarrow G/H$ and we have $\pi_1(G/H) \simeq H$ because $\pi \circ f(g) = Hf(g) = Hg$ for $f \in \text{Aut}(G)$, so $f(g) \simeq g$ implies that $f(g).g^{-1} \in H$ and $f(g) = h.g$, for some unique $h \in H$. Therefore $f = L_h$ and then we define $\text{Aut}(G) \rightarrow H$; $f \mapsto h$, which is an isomorphism. So, we may write $F: \pi_1(G/H) \rightarrow \text{Diff}(G)$ and $\Phi: \pi_1(G/H) \times G \rightarrow G$. The map $\Psi: H \times G \times G \rightarrow G \times G$ given by $\Psi(h, g_1, g_2) = (L_h(g_1), L_h(g_2))$ is a properly discontinuous action and defines a quotient manifold $V = \frac{G \times G}{\Psi}$, which equivalence classes are the orbits of Ψ . We have the following facts:

- 1) There exists a fibration $\sigma: V \rightarrow G/H$ with fiber G , induced by $\pi: G \rightarrow G/H$, and structural group isomorphic to $F(H) < \text{Diff}(G)$.
- 2) The natural foliation \mathcal{F} on G given by classes Hg ; $g \in G$, is Φ -invariant, such that the product foliation $G \times \mathcal{F}$ on $G \times G$ is Ψ -invariant and induces a foliation \mathcal{F}_0 on V , called suspension of \mathcal{F} for Φ , transverse to $\sigma: V \rightarrow G/H$.

2. Let $G = \mathbb{P}SL(2, \mathbb{C})$ and $H = \text{Aff}(\mathbb{C}) \triangleleft G$. An element of G has the expression $x \mapsto \frac{ax+b}{cx+d} = \frac{a+\frac{b}{x}}{c+\frac{d}{x}}$. The group H is the isotropy group of ∞ has its maps given by $x \mapsto \frac{ax+b}{d} \simeq \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix}$. Since $H \triangleleft G$, G has dimension 3 and H has dimension 2, we conclude that G/H has dimension one. Thus we have a fibration $\mathbb{P}SL(2, \mathbb{C}) \rightarrow \mathbb{CP}^1 \simeq S^2$ which is invariant under an action of $\text{Aff}(\mathbb{C})$ on $\mathbb{P}SL(2, \mathbb{C})$ having leaves diffeomorphic to $\mathbb{C}^* \times \mathbb{C}$.

The examples above are G -foliations which are also invariant under a Lie group transverse action in the sense of Definition 1, Section 5. As for the singular case, we observe that a one-dimensional logarithmic foliation on a manifold V^m as defined

in the introduction, is a G -foliation where G is the additive Lie group in \mathbb{C}^{m-1} , as granted by Theorem 3, Section 2.

4 G -Foliations with singularities

Let now \mathcal{F} be a one-dimension holomorphic foliation with isolated singularities on V . We shall say that \mathcal{F} admits a G -transverse structure if the corresponding nonsingular foliation $\mathcal{F}_0 := \mathcal{F}|_{V \setminus \text{sing}(\mathcal{F})}$ admits a G -transverse structure. Let $p \in \text{sing}(\mathcal{F})$ be an isolated singularity. Take a small neighborhood $p \in \Delta_p \subset V$ of p such that $\Delta_p \cap \text{sing}(\mathcal{F}) = \{p\}$ and Δ_p is biholomorphic to a ball in \mathbb{C}^m . Then $\Delta_p^* = \Delta_p \setminus \{p\}$ is simply-connected and by the classical theory of transversely Lie foliations (cf. Darboux-Lie Theorem 3) the restriction $\mathcal{F}|_{\Delta_p^*}$ is given by a holomorphic submersion $P: \Delta_p^* \rightarrow G$. By Hartogs' extension theorem ([11], [12]) the map extends to a holomorphic map $P: \Delta_p \rightarrow G$, which is a *first integral* for \mathcal{F} in Δ_p . Assume now that \mathcal{F} admits a G -transverse structure on $V \setminus \Lambda$ where $\Lambda \cap \text{sing}(\mathcal{F}) = \emptyset$ and $\Lambda \subset V$ is a codimension ≥ 2 analytic subset. By classical Hartogs' extension theorem the one-forms $\Omega_1, \dots, \Omega_\ell$ obtained in Theorem 3 extend holomorphically to V and then also by Theorem 3 the G -transverse structure extends to V . Hence we shall consider the case where \mathcal{F} admits a G -transverse structure on $V \setminus \Lambda$ and $\Lambda \subset V$ is analytic of codimension one. Also we shall assume that Λ is invariant by \mathcal{F} .

Lemma 1. *Let $X = \sum_{j=1}^m \lambda_j x_j \frac{\partial}{\partial x_j}$ in a neighborhood U of $0 \in \mathbb{C}^m$, with $\{\lambda_1, \dots, \lambda_m\}$ linearly independent over \mathbb{Q} . Denote by \mathcal{F}_X the one-dimensional foliation induced by X . Suppose there are holomorphic one-forms $\Omega_1, \dots, \Omega_{m-1}$ defined in $U_0 := U \setminus [\bigcup_{j=1}^m (x_j = 0)]$ such that:*

- i. $d\Omega_k = \sum_{i < j} c_{ij}^k \Omega_i \wedge \Omega_j$ where $\{c_{ij}^k\}$ are the structure constants of a Lie algebra of a Lie group G .
- ii. The foliation \mathcal{F}_X induced by X is given in U_0 by the integrable system $\{\Omega_1, \dots, \Omega_{m-1}\}$ of maximal rank.

Then

- 1. $d\Omega_j = 0, \forall j$.
- 2. Ω_j extends meromorphic to U as a closed one-form with simple poles.

Proof. For simplicity we assume $m = 3$. Let Θ_1, Θ_2 be given by $\Theta_j = \sum_{k=1}^3 a_k^j \frac{dx_k}{x_k}$, $a_k^j \in \mathbb{C}$. Then $\Theta_j(X) = \sum_{k=1}^3 a_k^j \lambda_k$. Thus we can choose Θ_1, Θ_2 such that these are linearly

independent in the complement of $\bigcup_{j=1}^3 (x_j = 0)$ and $\Theta_j(X) = 0, j = 1, 2$. Therefore \mathcal{F}_X is defined by the integrable system $\{\Theta_1, \Theta_2\}$ in U . By this and (ii) we can write in U_0

$$\Omega_1 = a_1\Theta_1 + a_2\Theta_2, \quad \Omega_2 = b_1\Theta_1 + b_2\Theta_2$$

for some holomorphic functions a_1, a_2, b_1, b_2 in U_0 with the property that $a_1b_2 - b_1a_2$ has no zero in U_0 . Now, since Θ_j is closed we have

$$d\Omega_1 = da_1 \wedge \Theta_1 + da_2 \wedge \Theta_2, \quad d\Omega_2 = db_1 \wedge \Theta_1 + db_2 \wedge \Theta_2$$

Thus $d\Omega_1 \wedge \Theta_1 = (da_1 \wedge \Theta_1 + da_2 \wedge \Theta_2) \wedge \Theta_1 = da_2 \wedge \Theta_2 \wedge \Theta_1$. From $d\Omega_1 = c_{12}^1 \Theta_1 \wedge \Theta_2$ we obtain $d\Omega_1 = c_{12}^1(a_1b_2 - a_2b_1)\Theta_1 \wedge \Theta_2 \wedge \Theta_1 = 0$. Thus we obtain

$$da_2 \wedge \Theta_1 \wedge \Theta_2 = 0$$

We have

$$\Theta_1 \wedge \Theta_2 = (a_1^1 a_2^2 - a_1^2 a_2^1) \frac{dx_1 \wedge dx_2}{x_1 x_2} + (a_1^1 a_3^2 - a_1^2 a_3^1) \frac{dx_1 \wedge dx_3}{x_1 x_3} + (a_2^1 a_3^2 - a_2^2 a_3^1) \frac{dx_2 \wedge dx_3}{x_2 x_3}$$

Write $\Theta_1 \wedge \Theta_2 = \alpha_{12} \frac{dx_1 \wedge dx_2}{x_1 x_2} + \alpha_{13} \frac{dx_1 \wedge dx_3}{x_1 x_3} + \alpha_{23} \frac{dx_2 \wedge dx_3}{x_2 x_3}$. For a holomorphic function $f(x_1, x_2, x_3)$ in U_0 we have

$$df \wedge \Theta_1 \wedge \Theta_2 = \left[\frac{f_{x_1} \alpha_{23}}{x_2 x_3} - \frac{f_{x_2} \alpha_{13}}{x_1 x_3} + \frac{f_{x_3} \alpha_{12}}{x_1 x_2} \right] dx_1 \wedge dx_2 \wedge dx_3$$

Therefore $df \wedge \Theta_1 \wedge \Theta_2 = 0 \Leftrightarrow \alpha_{23} x_1 f_{x_1} - \alpha_{13} x_2 f_{x_2} + \alpha_{12} x_3 f_{x_3} = 0$.

Now, if we write in Laurent series $f = \sum_{i,j,k \in \mathbb{Z}} f_{ijk} x_1^i x_2^j x_3^k$ then the last equation is equivalent to

$$\sum_{i,j,k \in \mathbb{Z}} (i\alpha_{23} - j\alpha_{13} + k\alpha_{12}) f_{ijk} x_1^i x_2^j x_3^k = 0$$

which is equivalent to

$$(i\alpha_{23} - j\alpha_{13} + k\alpha_{12}) f_{ijk} = 0, \quad i, j, k \in \mathbb{Z}$$

Recall that

$\alpha_{12} = a_1^1 a_2^2 - a_1^2 a_2^1$, $\alpha_{13} = a_1^1 a_3^2 - a_1^2 a_3^1$, $\alpha_{23} = a_2^1 a_3^2 - a_2^2 a_3^1$ and that we have $a_1^1 \lambda_1 + a_2^1 \lambda_2 + a_3^1 \lambda_3 = 0$, $a_1^2 \lambda_1 + a_2^2 \lambda_2 + a_3^2 \lambda_3 = 0$, so that the complex vectors $\vec{\alpha} := (\alpha_{12}, \alpha_{13}, \alpha_{23})$ and $\vec{\lambda} := (\lambda_1, \lambda_2, \lambda_3)$ are collinear, i.e., there is $t \in \mathbb{C}^*$ such that $\alpha_{12} = t\lambda_1$, $\alpha_{13} = t\lambda_2$, $\alpha_{23} = t\lambda_3$. Thus we have $df \wedge \Theta_1 \wedge \Theta_2 = 0 \Leftrightarrow (i\lambda_1 - j\lambda_2 + k\lambda_3) = 0, \forall i, j, k \in \mathbb{Z}$.

By the nonresonance hypothesis, the only solution to the last equation is the trivial solution, therefore such function f must be constant. This implies that a_2 is constant. Similarly a_1, b_1 and b_2 are constant in U_0 and we have

$$\begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = C \cdot \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}$$

for some nonsingular 2×2 complex matrix C . This proves (1) and (2) in the lemma. \square

Lemma 2. *Let \mathcal{F} be a one-dimensional holomorphic foliation with singularities on V^m . Assume that:*

1. *The singularities of \mathcal{F} are linearizable without resonances.*
2. *There is an invariant codimension one analytic subset $\Lambda \subset V^m$ such that \mathcal{F} has a G -transverse structure in $V^m \setminus \Lambda$.*

Then \mathcal{F} is a logarithmic foliation on V^m .

Proof. According to Darboux-Lie Theorem 3 there is a system of holomorphic one-forms $\Omega_1, \dots, \Omega_{m-1}$ defined in $V_0 := V \setminus \Lambda$ such that:

- i. $d\Omega_k = \sum_{i < j} c_{ij}^k \Omega_i \wedge \Omega_j$ where $\{c_{ij}^k\}$ are the structure constants of a Lie algebra of the Lie group G .
- ii. The foliation \mathcal{F} is given in V_0 by the integrable system $\{\Omega_1, \dots, \Omega_{m-1}\}$ of maximal rank.

Applying now Lemma 1 and denoting by η_j the meromorphic extension of Ω_j to V^m , we conclude that \mathcal{F} is given by a system of closed meromorphic one-forms $\{\eta_1, \dots, \eta_{m-1}\}$ on V^m , such that each η_j is holomorphic on $V^m \setminus \Lambda$ and has simple poles (contained on Λ). Therefore \mathcal{F} is a logarithmic foliation. \square

Remark 1. We can prove a more general version of the above lemma as follows: *Let \mathcal{F} be a one-dimensional foliation on \mathbb{CP}^m . Suppose that \mathcal{F} is given by a system of $m - 1$ meromorphic one-forms, η_j , such that each η_j is closed. Assume that each singularity of \mathcal{F} is linearizable without resonances. Then \mathcal{F} is logarithmic.*

Proof. Let $\Lambda \subset \mathbb{CP}^m$ be an irreducible component of the polar set $(\eta_j)_\infty$ of η_j . We claim that $\Lambda \cap \text{sing}(\mathcal{F}_X) \neq \emptyset$. Indeed, since Λ is an irreducible component of $(\eta_j)_\infty$ and η_j is closed it follows that Λ is invariant by the codimension one foliation induced by η_j and therefore Λ is invariant by \mathcal{F} . The Index Theorem in [19] and its natural generalizations in [20], then imply that necessarily $\Lambda \cap \text{sing}(\mathcal{F}) = \emptyset$. Given any singular point $p \in \text{sing}(\mathcal{F}) \cap \Lambda$ by hypothesis \mathcal{F} is linearizable without resonance at p and by the proof of Lemma 1 we conclude that η_j has simple poles in a neighborhood of p . This implies that Λ has order one in $(\eta_j)_\infty$. \square

From now on we assume that $V^m = \mathbb{CP}^m$ and \mathcal{F} is a one-dimensional foliation as in Theorem 1. Choose an affine space $\mathbb{C}^m \subset \mathbb{CP}^m$ such that the (projective) hyperplane $E_\infty := \mathbb{CP}^m \setminus \mathbb{C}^m$ is in general position with respect to \mathcal{F} , what means the following:

- $E_\infty \cap \text{sing}(\mathcal{F}) = \emptyset$.
- E_∞ is transverse to \mathcal{F} except for a finite number of tangency points.

- E_∞ meets each irreducible component of Λ transversely and at non-tangency points of \mathcal{F} .

Given affine coordinates $(z_1, \dots, z_m) \in \mathbb{C}^m = \mathbb{CP}^m \setminus E_\infty$, choose irreducible polynomials $f_1, \dots, f_s \in \mathbb{C}[z_1, \dots, z_m]$ such that $\Lambda = \bigcup_{j=1}^s (f_j = 0)$. Then, since each η_j is meromorphic on \mathbb{CP}^m , it is a rational one-form. Moreover, η_j is holomorphic on $\mathbb{CP}^m \setminus \Lambda$ and has only simple poles. The following result is well-known as the *Integration Lemma* (cf. [16] for instance):

Lemma 3. *We can write $\eta_j|_{\mathbb{C}^m} = \sum_{k=1}^s a_k^j \frac{df_k}{f_k}$ for some complex numbers $a_k^j \in \mathbb{C}$.*

Using this writing we can prove:

Lemma 4. *Given $j_1, \dots, j_r \in \{1, \dots, s\}$ the hypersurfaces $\{f_{j_1} = 0\}, \dots, \{f_{j_r} = 0\}$ intersect transversely.*

Proof. Indeed, given any point $p \in \{f_{j_1} = 0\} \cap \dots \cap \{f_{j_r} = 0\}$ of non transverse intersection, we have $df_{j_1}(p) \wedge \dots \wedge df_{j_r}(p) = 0$ and then the codimension-one foliations \mathcal{F}_{η_j} , defined by the one-forms η_j , $j \in \{j_1, \dots, j_r\}$ are not transverse at p . *A fortiori*, this implies that \mathcal{F} has a singularity at p . However, as we have seen in the proof of Lemma 1 the manifolds $\{f_j = 0\}$ are the separatrices of \mathcal{F} at p and therefore they are transverse at p . □

Given a singularity $p \in \text{sing}(\mathcal{F})$, because the irreducible components of Λ through p contain the local invariant hypersurfaces of \mathcal{F} passing through p , we conclude that $s \leq m$. We claim that $s = m$. Indeed, thanks to the local linearization, \mathcal{F} exhibits m local invariant analytic hypersurfaces through p , and each of these contained in some irreducible component Λ_j , $j \in \{1, \dots, s\}$ of Λ , as in the proof of Lemma 1. What we have to prove is that there exists no irreducible component Λ_j containing two different local invariant hypersurfaces passing through p . This is a kind of no self-connection for p in the real dynamics framework. In our case it will be a consequence of the nonresonance hypothesis. Choose a polynomial vector field X with isolated singularities on \mathbb{C}^m , such that $\mathcal{F}|_{\mathbb{C}^m}$ is induced by X and in particular $\text{sing}(\mathcal{F}) = \text{sing}(\mathcal{F}) \cap \mathbb{C}^m = \text{sing}(X)$. Given a singularity $p \in \text{sing}(\mathcal{F})$, since the irreducible components of Λ through p contain the local separatrices of \mathcal{F} through p , we conclude that if some irreducible component $\Lambda_j \ni p$ contains two different local invariant hypersurfaces through p then, because the residue of a closed meromorphic one-form is constant along an irreducible component of its polar set, we have two different local invariant hypersurfaces through p for which the residues of the forms $\eta_1, \dots, \eta_{m-1}$ are the same. Since, as in the proof of Lemma 1, there is a linear relation involving the eigenvalues of $DX(p)$ and the residues of the one-forms η_j along the irreducible components of Λ through p , we conclude that the eigenvalues of $DX(p)$ exhibit some resonance, contradiction. Thus we have $s = m$ and also:

Lemma 5. *If $\#[\text{sing}(\mathcal{F}) \cap \mathbb{C}^m] = 1$ then f_1, \dots, f_m are degree one polynomials.*

Proof. In fact, since these polynomials are transverse and $\text{sing}(\mathcal{F}) = \{f_1 = 0\} \cap \dots \cap \{f_m = 0\} \subset \mathbb{C}^m$ we have $\#[\text{sing}(\mathcal{F}) \cap \mathbb{C}^m] = \prod_{j=1}^m \deg(f_j)$ so that $\deg(f_j) = 1, \forall j = 1, \dots, m$. □

By an affine change of coordinates we can assume that the singularity $p = 0 \in \mathbb{C}^m$ is the origin. The following lemma is adapted from [13] and [5]:

Lemma 6. *Let X be a polynomial vector field with isolated singularities on $\mathbb{C}^m, m \geq 3$. Suppose that $\eta_j(X) = 0, j = 1, \dots, m-1$ where $\{\eta_1, \dots, \eta_{m-1}\}$ is a system of closed meromorphic one-forms, with simple poles, of maximal rank in the complement of $\bigcup_{j=1}^{m-1} (\eta_j)_\infty$ and that the corresponding one-dimensional foliation \mathcal{F} has a linearizable nonresonant singularity at $p \in \mathbb{C}^m$. Then X is a linear vector field in some affine chart in \mathbb{C}^m .*

Proof. Let $\Lambda_1, \dots, \Lambda_m$ be the irreducible components of $\Lambda \cap \mathbb{C}^m$ and the irreducible polynomials $f_1, \dots, f_m \in \mathbb{C}[z_1, \dots, z_m]$ such that $\Lambda_j = \{(z_1, \dots, z_m) \in \mathbb{C}^m : f_j(z_1, \dots, z_m) = 0\}, j = 1, \dots, m$. The polynomials f_j have degree one, vanish at the origin $0 = \text{sing}(X)$ of \mathbb{C}^m and intersect transversely at every point. Thus, we can find an affine change of coordinates $T(z_1, \dots, z_m) = (y_1, \dots, y_m)$ on \mathbb{C}^m such that $f_j(y_1, \dots, y_m) = y_j, j = 1, \dots, m$. The one-forms $\eta_j (j = 2, \dots, m)$ write as $\eta_j = \sum_{k=1}^m \alpha_k^j \cdot \frac{dy_k}{y_k}$ for some $\alpha_k^j \in \mathbb{C}$ and $X = \sum_{j=1}^m A_j(y_1, \dots, y_m) \frac{\partial}{\partial y_j}$ is polynomial such that $\eta_j(X) \equiv 0, \forall j = 2, \dots, n$. Since the hyperplanes $\{y_j = 0\} = \Lambda_j$ are X -invariant we must have $A_j(y_1, \dots, y_m) = y_j \cdot B_j(y_1, \dots, y_m)$ for some polynomials $B_j \in \mathbb{C}[y_1, \dots, y_m]$. Thus $X = \sum_{j=1}^m y_j \cdot B_j(y_1, \dots, y_m) \frac{\partial}{\partial y_j}$. Hence $0 = \eta_j(X) = \sum_{k=1}^m \alpha_k^j \cdot B_k(y_1, \dots, y_m), \forall j \in \{2, \dots, m\}$. Therefore $\sum_{k=1}^m \alpha_k^j \cdot B_k(0, \dots, 0) = 0, \forall j \in \{2, \dots, m\}$. Let then $Z := \sum_{k=1}^m B_k(0, \dots, 0) y_k \frac{\partial}{\partial y_k}$ be a linear vector field. By construction we have $\eta_j(Z) = 0, \forall j \in \{2, \dots, n\}$. Therefore the foliations $\mathcal{F}(X)$ defined by X and \mathcal{F}_Z defined by Z coincide on \mathbb{C}^m and thus $X = H \cdot Z$ for some polynomial H on \mathbb{C}^m . Since $\text{sing}(X) = \{0\} = \text{sing}(Z)$ the polynomial H has no zeros on \mathbb{C}^m and it is therefore constant say $H = \lambda \in \mathbb{C}^*$. Thus we have proved that X is linear in the variables $(y_1, \dots, y_m) \in \mathbb{C}^m$ of the form

$$X(y_1, \dots, y_m) = \lambda \cdot \sum_{k=1}^m B_k(0, \dots, 0) y_k \frac{\partial}{\partial y_k}.$$

This ends the proof. □

The proof of Theorem 1 follows from the above lemmas. Theorem 2 follows from (the proof of) Lemma 2.

5 Foliations invariant under G -transverse actions

Let V be a manifold, \mathcal{F} a codimension ℓ foliation on V and G a Lie group of dimension $\dim G = \text{codim } \mathcal{F} = \ell$.

Definition 1 ([3]). We say that \mathcal{F} is *invariant under a transverse action* of the group G , \mathcal{F} is G -i.u.t.a. for short, if there is an action $\Phi: G \times V \rightarrow V$ of G on V such that: (i) the action is *transverse to \mathcal{F}* , i.e., the orbits of this action have dimension ℓ and intersect transversely the leaves of \mathcal{F} and (ii) Φ *leaves \mathcal{F} invariant*, i.e., the maps $\Phi_g: x \mapsto \Phi(g, x)$ take leaves of \mathcal{F} onto leaves of \mathcal{F} .

Let \mathcal{F} be a foliation on V such that \mathcal{F} is G -i.u.t.a. It is not difficult to prove the existence of a *Lie foliation structure for \mathcal{F} on V of model G* in the sense of Ch. III, Sec. 2 of [10]. We shall then say that \mathcal{F} *has G -transversal structure* and prove (with a self-contained proof) the existence of a *development* for \mathcal{F} as in Proposition 2.3, page 153 of [10] (Ch. III, Sec. 2). A sort of strong form of this procedure can be found in [3], Section 4 with a self-contained proof (Proposition 3).

In this section all foliations are assumed to be holomorphic, though our arguments hold for class C^∞ . First of all we remark the existence of a Lie transverse structure for the foliation invariant under a Lie group transverse action:

Proposition 1. *Let V^m be a manifold equipped with a codimension ℓ foliation \mathcal{F} invariant under a transverse action of a Lie group G . Then \mathcal{F} has a Lie transverse structure of model G . Indeed, there exists an integrable system $\{\Omega_1, \dots, \Omega_\ell\}$ of one-forms defining \mathcal{F} on V with $d\Omega_k = \sum_{i < j} c_{ij}^k \Omega_i \wedge \Omega_j$, where $\{c_{ij}^k\}$ are the structure constants of the Lie algebra of G .*

Proof. Let Φ a transverse action of G on a manifold V of dimension m , \mathcal{F} a codimension ℓ foliation invariant under the Lie group transverse action Φ , $\{X_1, \dots, X_\ell\}$ a basis of Lie algebra of G and $\{\omega_1, \dots, \omega_\ell\}$ the corresponding dual basis. We have $d\omega_k = \sum_{i < j} c_{ij}^k \omega_i \wedge \omega_j$. We consider an open cover $\{U_i\}$ of V by coordinate systems of \mathcal{F} : given by local charts $\phi_i = (\alpha_i, \beta_i): U_i \subset V \rightarrow D_1^i \times D_2^i \subset \mathbb{C}^{m-\ell} \times \mathbb{C}^\ell$, and the projection $\pi_2: \mathbb{C}^{m-\ell} \times G \rightarrow G$ on the second factor. Given any point $x \in V$ we denote by \mathcal{F}_x the leaf of \mathcal{F} through x . Then each map $\pi_2 \circ \phi_i$ defined by $\pi_2 \circ \phi_i(x) = \beta_i(x)$ is a submersion and the plaques of \mathcal{F} in U_i are given by the intersections $\mathcal{F}_x \cap U_i = \phi_i^{-1}(D_1^i \times \{\beta_i(x)\}) = (\pi_2 \circ \phi_i)^{-1}(\beta_i(x))$, that is, the restriction $\mathcal{F}|_{U_i}$ is given by $\pi_2 \circ \phi_i = \text{constant}$. Thus, if $U_i \cap U_j \neq \emptyset$ then there exists a locally constant map $\gamma_{ij}: U_i \cap U_j \rightarrow \{\text{left translations on } G\}$ such that $\pi_2 \circ \phi_i(x) = \gamma_{ij}(x)(\pi_2 \circ \phi_j(x))$ for all $x \in U_i \cap U_j$. Now, given a point $p \in V$, there exists a local chart $(U, \phi = (\alpha, \beta))$ of \mathcal{F} with $\phi: U \rightarrow D_1 \times D_2 \subset \mathbb{C}^{m-\ell} \times G$, $\phi(p) = (0, e)$ and ϕ takes every leaf of $\mathcal{F}|_U$ to a fiber $\mathbb{C}^{m-\ell} \times \{\text{constant}\}$ of $\mathbb{C}^{m-\ell} \times G$. We may write $\phi(\Phi_g(x)) = (\xi(x, g), g)$, for all (x, g) close to $(0, e)$.

We define one-forms in $\Omega_1, \dots, \Omega_\ell$ in V by setting in U the definition $\Omega_1 = (\pi_2 \circ \phi)^* \omega_1, \dots, \Omega_\ell = (\pi_2 \circ \phi)^* \omega_\ell$. Since ω_i is left-invariant (that is, $L_g^* \omega_i = \omega_i, \forall g \in G$) we conclude that Ω_i is well defined. In U we have $\Omega_k = (\pi_2 \circ \phi)^* \omega_k = (\pi_2 \circ \phi)^* d\omega_k$ and

then $d\Omega_k = (\pi_2 \circ \phi)^*(\sum_{i < j} c_{ij}^k \omega_i \wedge \omega_j) = \sum_{i < j} c_{ij}^k (\pi_2 \circ \phi)^* \omega_i \wedge (\pi_2 \circ \phi)^* \omega_j = \sum_{i < j} c_{ij}^\nu \Omega_i \wedge \Omega_j$. This shows that $d\Omega_\nu \wedge \Omega_1 \wedge \dots \wedge \Omega_\ell = 0$, because $c_{ij}^\nu = c_{ji}^\nu$ and then, by Frobenius' Theorem [2], the system $\{\Omega_1, \dots, \Omega_\ell\}$ defines an integrable field of ℓ -plans \mathcal{S} . The condition $T_p \mathcal{F} \oplus T_p \Phi_p(G) = T_p V$ implies that any vector, tangent to \mathcal{F} , has no projection over G . We have $d\pi_2(x, y) = \begin{pmatrix} 0 & 0 \\ 0 & \text{Id} \end{pmatrix} \in M_{\ell \times m}(\mathbb{C})$, so $d\pi_2(x, y)(v, u) = u$, that is, $d\pi_2(x, y)|_{\mathbb{C}^{m-\ell} \times \{0\}} \equiv 0$. For a tangent vector $v = (X, 0)$ on \mathcal{F}_p we have $\Omega_j(p)(X, 0) = (\pi_2 \circ \Phi)^* \omega_j(p)(X, 0) = \omega_j(\pi_2 \circ \Phi(p))d(\pi_2 \circ \Phi)(p)(X, 0) = 0$ because $d(\Phi(p))(X, 0) \in \mathbb{C}^{m-\ell} \times \{0\}$. Hence, every one-form Ω_j is zero along the leaves of \mathcal{F} and, since $\dim \mathcal{F} = \dim \mathcal{S}$, we conclude that $\mathcal{F} = \mathcal{S}$. \square

Remark 2. Proposition 1 above can be proved using the fact that the connection tangent to a foliation is flat and therefore its Cartan's curvature form is zero (see [3]), nevertheless we have chosen to give explicit constructive arguments which we think may be useful for an eventual consideration of the singular case.

Proposition 1 and Theorem 1 then promptly give:

Corollary 1. *Let \mathcal{F} be a one-dimensional holomorphic foliation with singularities on \mathbb{CP}^m . Assume that:*

1. *The singularities of \mathcal{F} are linearizable without resonances.*
2. *There is an invariant codimension one algebraic subset $\Lambda \subset \mathbb{CP}^m$ such that \mathcal{F} is invariant under a G -transverse action in $\mathbb{CP}^m \setminus \Lambda$.*

Then \mathcal{F} is a logarithmic foliation. If moreover $\sharp[\text{sing}(\mathcal{F}) \cap \mathbb{C}^m] = 1$ for an affine space $\mathbb{C}^m \subset \mathbb{CP}^m$ such that $\mathbb{CP}^m \setminus \mathbb{C}^m$ is in general position with respect to \mathcal{F} then \mathcal{F} is linearizable.

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